On the spectrum of QCD_{1+1} with large numbers of flavours N_F and colours N_C near $N_F/N_C = 0$

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QCD_{1+1} in the limit of a large number of flavours N_F and a large number of colours N_C is examined in the small N_F/N_C regime. Using perturbation theory in N_F/N_C, stringent results for the leading behaviour of the spectrum departing from $N_F/N_C = 0$ are obtained. These results provide benchmarks in the light of which previous truncated treatments of QCD_{1+1} at large N_F and N_C are critically reconsidered.

PACS: 11.10.Kk, 11.15.Pg, 12.38.-t
Keywords: QCD_{1+1}, large number of flavours, large number of colours

I. INTRODUCTION

QCD in one space and one time dimension with large numbers of flavours N_F and colours N_C has repeatedly attracted attention [1]- [7], for various reasons. For one, the success of the $1/N_C$ expansion provokes the question as to which of its aspects are modified if also the number of flavours N_F is taken to be large; after all, while one may arguably regard $1/N_C$ as a small number in the real world, this is much harder to justify for N_F/N_C. Compared with the one flavour, many colour case [8], which is exactly solvable for $N_C \to \infty$ and yields a Regge trajectory of mesons consisting strictly of two partons, the many flavour model is considerably more complex. The suppression of quark-antiquark pair creation engendered by the large N_C limit [9] is offset by the increasing number of vacuum polarisation options as N_F becomes large. In this way, a nontrivial limit of $N_C \to \infty$ at constant N_F/N_C emerges which allows for a much wealthier spectrum than the one-flavour limit. While quark exchange interactions between different flavour singlet excitations are still suppressed [3], pair creation and annihilation effects within flavour singlets lead to these particles becoming complicated mixtures of states of different parton number. They resemble chains whose links are characterized by the quark and the antiquark at either end of the link being, alternatingly, either coupled to a colour or a flavour singlet. In this respect, there is a correspondence to QCD_{1+1} with one flavour, but adjoint colour quarks; also in the latter model, extended chain structures form due to each quark possessing two fundamental colour indices which need to be saturated. This not only shows up in large N_C studies [10]- [14], but also in the thermodynamical behaviour at $N_C = 2$, cf. [15]. The correspondence between the multiflavour, fundamental colour models and the one-flavour adjoint colour models can in fact be put on a formally exact footing for $N_F = N_C \to \infty$, where the massive spectra in the massless quark case can be shown to coincide [16].

Not least due to this correspondence, there have been repeated numerical studies of QCD_{1+1} at large N_F and N_C; these include ones using quark degrees of freedom [3] as well as more recent and elaborate ones within the bosonization framework [6]. However, all these investigations use more or less uncontrolled truncations of the theory. The purpose of the present note is to assess the severity of these truncations by comparing with results obtained in a particular regime which is under good numerical control, namely the regime of small N_F/N_C. More specifically, the leading behaviour, proportional to N_F/N_C, departing from the 't Hooft meson spectrum [8] at $N_F/N_C = 0$ is obtained using a simple perturbative calculation. Some agreement, but also significant discrepancies are found as compared with the aforementioned studies, even the more recent ones. This indicates that, at this stage, these treatments are not yet as reliable numerically as one might hope for. The results reported here should provide valuable benchmarks for future more elaborate studies of QCD_{1+1} in the limit of large N_F and N_C.

II. FOCK SPACE AND PERTURBATION THEORY

The following treatment will concentrate on the case of zero quark mass matrix, which maximally allows for the pair creation and annihilation effects highlighted further above. Light-cone coordinates are used and, moreover, only

*Supported by DFG under contract En 415/1–1.

the sector of overall flavour singlet states will be considered. In view of colour confinement, physical states will be composed out of (colour singlet) 't Hooft mesons [8], which are created by the quark bilinear operators

$$M^\dagger_{Qnab} = \frac{1}{\sqrt{N_C}} \frac{1}{\sqrt{Q}} \int_0^Q dp\, \phi_n(p/Q) \sum_i q^\dagger_{ai}(p) q_{bi}(p-Q) \tag{1}$$

where Q denotes the light-cone momentum of the meson, n its excitation number, i is the colour index, and a, b the flavour indices. The wave functions ϕ_n satisfy 't Hooft's eigenvalue equation[†]

$$-\left(\frac{1}{x} + \frac{1}{1-x}\right) \phi_n(x) - \int_0^1 \frac{dy}{(x-y)^2} \phi_n(y) = \frac{2\pi}{g^2 N_C} \mu_n^2 \phi_n(x) \; , \qquad x \in [0,1] \tag{2}$$

and form an orthonormal complete set of functions on the interval $[0,1]$ with the boundary conditions $\phi'(0) = \phi'(1) = 0$; the eigenvalues μ_n^2 represent the invariant square masses of the mesons. The singularity in the Coulomb propagator in (2) is defined as usual via the principal value prescription[‡]. By using the properties of quark operators acting on the vacuum, which is the perturbative one in light-cone quantization,

$$q^\dagger_{ai}(p)|0\rangle \quad \text{for } p < 0 \tag{3}$$

$$q_{ai}(p)|0\rangle \quad \text{for } p > 0 \tag{4}$$

one can convince oneself that the flavour singlet one-meson states

$$|2K, n\rangle = \frac{1}{\sqrt{N_F}} \sum_a M^\dagger_{(2K)naa} |0\rangle \tag{5}$$

are normalized as

$$\langle 2K', n' | 2K, n\rangle = \delta(2K - 2K') \delta_{nn'} \tag{6}$$

Likewise, one can define flavour singlet two-meson states[§], where one has two options of coupling the flavours:

$$|K + Q/2, m; K - Q/2, n\rangle_{SS} = \frac{1}{\sqrt{2} N_F} \sum_{a,b} M^\dagger_{(K+Q/2)maa} M^\dagger_{(K-Q/2)nbb} |0\rangle \tag{7}$$

$$|K + Q/2, m; K - Q/2, n\rangle_{NN} = \frac{1}{\sqrt{2} N_F} \sum_{a,b} M^\dagger_{(K+Q/2)mab} M^\dagger_{(K-Q/2)nba} |0\rangle \tag{8}$$

Either both of the mesons are flavour singlets by themselves (7), or two flavour non-singlets are coupled to an overall singlet (8). Furthermore, to avoid double-counting of states, a definite ordering of the meson momenta in the states will be adopted, namely $Q \geq 0$. This means that for $m \neq n$, one must distinguish between the states (7),(8) and the corresponding states with m and n exchanged. The singlet-singlet states (7) completely decouple from other states of lower or equal parton number in the large N_C limit [3], and need not be considered further here. The states (8) of overall momentum $2K$ are normalized as

$$\langle K' + Q'/2, m'; K' - Q'/2, n' | K + Q/2, m; K - Q/2, n\rangle = \delta(2K - 2K') \delta(Q - Q') \delta_{mm'} \delta_{nn'} \tag{9}$$

where the subscript NN on the states is dropped here and in the following. The light-cone Hamiltonian of QCD_{1+1} in the light-cone gauge reads

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[†]Note that the coupling constant g^2 used here and in [3] differs from the one used in [6] and [7] by a factor 2.

[‡]For a dynamical regularization of the Coulomb propagator via gauge field zero modes, cf. [17].

[§]Note that the normalization of these states differs from the one in [3] by a factor $1/\sqrt{2}$. There, this factor was instead absorbed into the relative momentum wave function of the two mesons. It arises when classifying states according to total and relative momenta as opposed to the individual momenta of the mesons.

$$
\begin{aligned}
H = &-\frac{g^2}{4\pi}\frac{N_C^2-1}{N_C}\sum_{a,i}\int\frac{dk}{k}:q_{ai}^\dagger(k)q_{ai}(k): \\
&-\frac{g^2}{8\pi}\sum_{a,b,i,j}\int\frac{dq}{q^2}dk\,dk'\,\Big(:q_{ai}^\dagger(k)q_{bi}(k'+q)q_{bj}^\dagger(k')q_{aj}(k-q): \\
&\qquad\qquad +\frac{1}{N_C}:q_{ai}^\dagger(k)q_{ai}(k-q)q_{bj}^\dagger(k')q_{bj}(k'+q):\Big)
\end{aligned}
\tag{10}
$$

It has been normal-ordered with respect to the perturbative vacuum. In the 't Hooft limit $N_C\to\infty$, $N_F/N_C\to 0$, the meson states defined further above become eigenstates of the Hamiltonian,

$$
H|2K,n\rangle = \frac{\mu_n^2}{4K}|2K,n\rangle \tag{11}
$$

$$
H|K+Q/2,m;K-Q/2,n\rangle = \left(\frac{\mu_m^2}{2K+Q}+\frac{\mu_n^2}{2K-Q}\right)|K+Q/2,m;K-Q/2,n\rangle \tag{12}
$$

When N_F/N_C remains finite as $N_C\to\infty$, the only non-vanishing matrix elements involving one-meson states are

$$
\langle 2K',n'|H|2K,n\rangle = \delta(2K-2K')\delta_{nn'}\frac{\mu_n^2}{4K} \tag{13}
$$

$$
\langle 2K',n'|H|K+Q/2,m;K-Q/2,n\rangle = \sqrt{\frac{N_F}{N_C}}\frac{g^2N_C}{16\pi K^{3/2}}(1-(-1)^{m+n+n'})f_{mnn'}\left(\frac{K+Q/2}{2K}\right)\delta(2K-2K') \tag{14}
$$

with the form factor**

$$
f_{mnn'}(v) = \frac{1}{\sqrt{v(1-v)}}\int_0^v dx\int_0^{1-v}dy\,\phi_m(x/v)\phi_n(y/(1-v))\frac{\phi_{n'}(x)-\phi_{n'}(v+y)}{(v+y-x)^2} \tag{15}
$$

With these expressions, it is now straightforward to derive the leading perturbative corrections, of order N_F/N_C, to the masses of the 't Hooft mesons. The perturbed eigenvalues $E(2K,n)$ of the Hamiltonian are given by

$$
\begin{aligned}
&E(2K,n)\langle 2K',n|2K,n\rangle = \\
&\frac{\mu_n^2}{4K}\langle 2K',n|2K,n\rangle \\
&+\int_0^\infty d(2\bar{K})\int_0^{2\bar{K}}dQ\sum_{\bar{m},\bar{n}=0}^{\infty}\frac{\langle 2K',n|H|\bar{K}+Q/2,\bar{m};\bar{K}-Q/2,\bar{n}\rangle\langle\bar{K}+Q/2,\bar{m};\bar{K}-Q/2,\bar{n}|H|2K,n\rangle}{\mu_n^2/4K\;-\;\mu_{\bar{m}}^2/(2\bar{K}+Q)\;-\;\mu_{\bar{n}}^2/(2\bar{K}-Q)}
\end{aligned}
\tag{16}
$$

cf. (12). This expression deserves some comment for meson excitation number $n=2$ and higher, since these states already lie above production thresholds for two-meson states containing massive mesons of lower n (the zeroth 't Hooft meson is massless). Therefore, degenerate perturbation theory is called for. In this respect, the cases $n=2$ through $n=4$ are distinct from $n\geq 5$:

The 't Hooft mesons of excitation number $n=2$ through $n=4$ lie in a continuum of two-meson states with the special property that one of the mesons in these states always carries excitation number zero, i.e. either $\bar{m}=0$ or $\bar{n}=0$ in (16). Now, the form factor (15) satisfies the relations

$$
f_{m0n}\left(\frac{\mu_m^2}{\mu_n^2}\right) = f_{0mn}\left(1-\frac{\mu_m^2}{\mu_n^2}\right) = 0 \tag{17}
$$

shown in the Appendix. Because of this, zeroes of the energy denominator in (16) are cancelled by zeroes of the coupling matrix elements; i.e., the perturbed Hamiltonian is already diagonal in the subspace of states (quasi-)degenerate with

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**In numerical evaluations of the form factor, the behaviour around the integration point $(x,y)=(v,0)$, at which the integrand superficially becomes singular, can be cast into a manifestly nonsingular form by using radial coordinates around this point.

the 't Hooft mesons in question. Therefore, these mesons represent legitimate unperturbed states of degenerate perturbation theory and (16) remains consistent.

By contrast, mesons of excitation number $n \geq 5$ are degenerate with two-meson states in which both mesons carry nonzero excitation number. In this case, (16) develops poles (this was verified numerically), and becomes inconsistent as it stands. Instead, the single-meson states in question must be mixed with (quasi-)degenerate higher parton number states already at the unperturbed level, such as to rediagonalize the Hamiltonian in this subspace. This more complex case will not be considered further here; (16) was only evaluated numerically for the 't Hooft mesons $n = 1$ through $n = 4$, which in view of the above discussion represent legitimate unperturbed eigenstates even in the presence of degeneracies with the two-meson continuum.

Inserting (14) into (16), one can cast the invariant square masses in terms of the dimensionless quantities $e_0(n)$ and $e_1(n)$,

$$4K \frac{2\pi}{g^2 N_C} E(2K, n) = e_0(n) + \frac{N_F}{N_C} e_1(n) \tag{18}$$

with

$$e_0(n) = \frac{2\pi}{g^2 N_C} \mu_n^2 \tag{19}$$

$$e_1(n) = \frac{1}{2} \sum_{\bar{m},\bar{n}=0}^{\infty} \int_0^1 dx \, \frac{(1-(-1)^{\bar{m}+\bar{n}+n})^2}{e_0(n) - 2e_0(\bar{m})/(1+x) - 2e_0(\bar{n})/(1-x)} \left(f_{\bar{m}\bar{n}n}\left(\frac{1+x}{2}\right)\right)^2 \tag{20}$$

The following table summarizes the results of the numerical evaluation of $e_0(n)$ and $e_1(n)$ for the 't Hooft mesons with excitation number $n = 1$ through $n = 4$.

n	1	2	3	4
$e_0(n)$	5.88	14.14	23.08	32.30
$e_1(n)$	5.1	12.0	-30.5	9.1

In the evaluation, the sums over $\bar{m}, \bar{n}$ were truncated at $\bar{m} = 10$, $\bar{n} = 10$; at this level of approximation, neglected terms are suppressed compared with the sums obtained by a factor 10^{-4} (in the case of $n = 4$, by a factor $4 \cdot 10^{-4}$). Furthermore, since all neglected terms are negative, the values given for $e_1(n)$ represent rigorous upper bounds for the exact values.

The results given in in the above table can be compared to results of previous truncated numerical studies of the model. Starting with the first massive state, the studies [6] and [7] agree on a slope $e_1(1) = 5$ as N_F/N_C is increased from zero, which is confirmed by the result $e_1(1) = 5.1$ obtained here. In [3], this state was discarded as uninteresting for the (not very compelling) reason that it exhibited a positive expectation value of the potential energy.

Considering further excited states, however, the spectrum displayed in [6] suggests that masses systematically rise as N_F/N_C is increased from zero[††]; indeed, it has been speculated [6] that N_F/N_C essentially acts as a mass in the model. This is in qualitative disagreement with the result obtained above, where in particular the trajectory associated with the third 't Hooft meson decreases very strongly[‡‡]. This disagreement is not entirely surprising in view of the delicate cancellation observed in connection with (16) for mesons with excitation number $n = 2$ and higher, which are embedded in a continuous spectrum of two-meson states. Already slight truncations in the numerical treatment will destroy this cancellation and will thus lead to a remixing of 't Hooft's mesons with the two-meson continuum already at the unperturbed level, such that results at the next order, i.e. N_F/N_C, become completely unreliable. This is certainly what happened in [3], and in view of the strong discrepancy between the result arrived at in the present work and the behaviour displayed in [6], it evidently also takes place there. Indeed, the state $n = 3$ already is claimed to be a strong mixture of different parton number states in [6].

On a more speculative level, the systematic rise of the spectral trajectories in the bosonization treatments [6], [7] may also be tied to the behaviour found in these studies at large N_F/N_C, where only invariant square masses of the order of $g^2 N_F$ are detected. It is tempting to conjecture that a rich spectrum at the lower scale $g^2 N_C$ is thus entirely discarded; such a spectrum would seem to arise naturally in a perturbative treatment similar in spirit to the one

[††]Note that [6] gives results only for the sector of states which become 't Hooft mesons of odd excitation number in the limit $N_F/N_C = 0$.

[‡‡]In this respect, the results of [3] display at least qualitatively correct behaviour.

presented above[§§]. At large N_F/N_C, it is initially indeed natural to measure energies in units of g^2N_F; consider e.g. recasting (13) and (14) in units of g^2N_F instead of g^2N_C. Then, formally, (13) is of the order N_C/N_F and (14) is of the order $\sqrt{N_C/N_F}$, whereas the coupling of two-meson states to two-meson states (given explicitly e.g. in [3]) is of order one[***]. Thus, in the large N_F/N_C limit, one would regard the latter as the unperturbed Hamiltonian and treat (13) and (14) as perturbations in N_C/N_F. Therefore, the unperturbed problem initially yields a spectrum of energies proportional to g^2N_F, as indeed happens in the bosonization approach [4]- [7]. However, this is only half the story; it should be realized that in this scheme there is a rich spectrum of (unperturbed) massless states. Among them are e.g. all of 't Hooft's mesons. Now, even if one is only interested in the massive spectrum of the model, one should not prematurely discard these zero energy states[†††], since their masses presumably are not protected against corrections once one includes the perturbations in N_C/N_F. Already at the next order, i.e. $g^2N_F \cdot N_C/N_F$, single meson states acquire corrections from two sources: First order perturbation theory in (13), yielding precisely 't Hooft's masses, and second order perturbation theory in (14), coupling to the two-meson states[‡‡‡]. Thus, in the limit $N_F/N_C \to \infty$, there is a sound basis for the conjecture that a well-defined spectrum of invariant square masses of the order of g^2N_C emerges.

In summary, it seems that more work is needed before a coherent picture of QCD_{1+1} with a large number of colours and flavours can be presented; the truncation schemes hitherto applied leave room for error even on a qualitative level. The results presented here for the first time provide stringent numerical control over the behaviour of the spectrum departing from the 't Hooft limit $N_F/N_C = 0$, albeit in a very limited range of the parameter N_F/N_C, and only for the low-lying 't Hooft mesons of excitation number $n = 1$ through $n = 4$. These results may provide useful benchmarks for future improved studies of the model.

APPENDIX

Consider the form factor $f_{m0n}(v)$, cf. eq. (15). Inserting the special form of the zeroth 't Hooft meson wave function, $\phi_0(x) = 1$, this form factor satisfies

$$\sqrt{v(1-v)}f_{m0n}(v) = \int_0^v dx \int_0^{1-v} dy\, \phi_m(x/v) \frac{\phi_n(x) - \phi_n(v+y)}{(v+y-x)^2} \tag{21}$$

$$= \int_0^v dx\, \phi_m(x/v)\phi_n(x) \int_0^{1-v} dy\, \frac{1}{(v+y-x)^2} \tag{22}$$

$$-v \int_0^1 dz \int_{-v}^{1-v} dy\, \frac{\phi_m(z)\phi_n(v+y)}{(v+y-vz)^2} + v \int_0^1 dz \int_{-v}^{0} dy\, \frac{\phi_m(z)\phi_n(v+y)}{(v+y-vz)^2} \tag{23}$$

$$= \int_0^v dx\, \phi_m(x/v)\phi_n(x) \left(\frac{1}{v-x} - \frac{1}{1-x} \right) \tag{24}$$

$$-v \int_0^1 dz \int_0^1 dy\, \frac{\phi_m(z)\phi_n(y)}{(y-vz)^2} + \frac{1}{v} \int_0^1 dz \int_0^v dy\, \frac{\phi_m(z)\phi_n(y)}{(y/v-z)^2} \tag{25}$$

Using 't Hooft's equation (2) to carry out the y-integral in the first term in (25) and, likewise, the z-integral in the second term in (25), one arrives at

$$\sqrt{v(1-v)}f_{m0n}(v) = v \int_0^1 dz\, \phi_m(z)\phi_n(vz) \left(\frac{1}{v-vz} - \frac{1}{1-vz} \right) \tag{26}$$

$$-v \int_0^1 dz\, \phi_m(z) \left(-\frac{1}{vz} - \frac{1}{1-vz} - \frac{2\pi}{g^2N_C}\mu_n^2 \right) \phi_n(vz) \tag{27}$$

[§§]A first, albeit strongly truncated, glimpse of a spectrum at the scale g^2N_C was seen in [3].

[***]Note that no higher orders arise because acting with the Hamiltonian on a multi-meson state only modifies up to two mesons in that state.

[†††]Note that the treatment of the large N_F limit in [5] invokes a saddle-point argument based on the magnitude of N_F; great care must be exercised in justifying such an argument in the presence of (quasi-) zero modes.

[‡‡‡]Of course, additional care must be taken if massless unperturbed two-meson states occur in this scheme.

$$+\frac{1}{v}\int_0^v dy\,\phi_n(y)\left(-\frac{v}{y}-\frac{1}{1-y/v}-\frac{2\pi}{g^2N_C}\mu_m^2\right)\phi_m(y/v) \tag{28}$$

$$=\int_0^1 dz\,\phi_m(z)\phi_n(vz)\left(\frac{1}{1-z}-\frac{v}{1-vz}+\frac{1}{z}+\frac{v}{1-vz}+\frac{2\pi}{g^2N_C}v\mu_n^2\right) \tag{29}$$

$$+\int_0^1 dz\,\phi_n(vz)\phi_m(z)\left(-\frac{1}{z}-\frac{1}{1-z}-\frac{2\pi}{g^2N_C}\mu_m^2\right) \tag{30}$$

$$=\frac{2\pi}{g^2N_C}\int_0^1 dz\,\phi_m(z)\phi_n(vz)(v\mu_n^2-\mu_m^2) \tag{31}$$

At the particular value $v=\mu_m^2/\mu_n^2$, the form factor $f_{m0n}(v)$ thus exhibits a zero. The case of $f_{0mn}(1-\mu_m^2/\mu_n^2)$ can be treated in complete analogy.

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